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Energy–momentum of the self-fields of a moving charge in classical electromagnetism

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Abstract. The fundamental problem of the energy and momentum of the self-fields of a moving charge in the classical theory of electromagnetism has not yet been solved to full satisfaction. The widely-held belief that the energy and momentum of the electromagnetic field of a moving charge should behave as components of a 4-vector under a Lorentz transformation, is not borne out by the conventional theory. This apparent anomaly has led to extensive attempts on reinterpretations or even to suggestions for outright modifications of some basic aspects of the classical theory of electromagnetism. We show here that such drastic steps are not actually needed and that the above mentioned belief is ill-founded. A relativistically consistent picture emerges in the conventional theory when a proper account is taken of all the energy and momentum associated with the electromagnetic phenomenon in the system.

1. Introduction

The classical theory of electromagnetism (CTEM) is generally accepted to be in conformity with the special theory of relativity. In fact, the Lorentz transformations for the electromagnetic (EM) fields were derived [1] even before Einstein put forward the special theory of relativity [2]. However, there is one aspect of the CTEM which appears to be non-compatible with the special theory of relativity. There are conceptual difficulties perceived within the CTEM when one tries to calculate the energy and momentum associated with the EM field of a *moving* charge, which make it almost appear as if in the conventional theory of classical electromagnetism, the concept of simple charged particles and electromagnetic fields are in some way inconsistent [3].

This problem has been known in the literature for a long time and its detailed history has been documented elsewhere [4–7]. We shall briefly mention some of it here, only to put the problem in a proper perspective. In 1881, Thomson [8] made the first attempt in calculating the electromagnetic contribution to the mass of a charged particle, by identifying the energy in the magnetic field of a moving charge with the kinetic energy of motion of its electromagnetic mass. In 1903, Abraham [9] proposed a purely EM model of the 'elementary' charged particle, namely the electron, which had then only recently been discovered by Thomson [10]. Abraham assumed that the mass of an elementary charge was purely of electromagnetic origin and he calculated

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the mass from the amount of momentum in the EM fields of the charge, when set in a uniform motion. In these calculations, Abraham always took the charge distribution to be spherically symmetric, irrespective of the motion of the charged particle. Lorentz [1] modified Abraham's calculations by proposing that during a motion 'through the ether', length of the charged particle would contract in the direction of its motion. The results thus derived by Lorentz remain valid even for a post-relativistic model, even though the reasons put forward by him for the 'Lorentz-contraction' are unacceptable. Subsequently Lorentz [11] also calculated the inertial mass of a charged particle, in a more sophisticated way, from its rate of change of momentum in the presence of its electromagnetic self-interaction. The inertial mass in all such calculations turned out to be $4U/3c^2$ (see for example, Schott [12] for detailed calculations), where U is the electrostatic self-potential energy of the charge distribution, also equal to the volume integral of the electrostatic field energy-density, and c is the speed of light in vacuum. With the advent of the special theory of relativity it became clear that there is a mass associated with all energies and that the expected value of mass due to self-fields would be U/c^2. This puzzling factor of 4/3 in the inertia of EM energy has ever since been highly annoying.

In 1906, Poincaré [13] pointed out that such a pure EM charge particle would be unstable due to forces of self-electrostatic repulsion. By postulating the presence of some 'unknown internal (negative) pressure' within the electron for its stability and by including an appropriate contribution from these non-electromagnetic stresses he was able to get the 'right' relations for the *total* energy and momentum. Since nothing was known about the origin and fundamental nature of these 'Poincaré stresses' within the electron, these *seemed* to be chosen merely to fit the solution. Further this made it look as if one could not even calculate the energy-momentum content of just EM fields, without questioning the stability of the associated charge-distribution, perhaps through some non-electromagnetic interactions, and as if the CTEM were not complete in itself. Over the time, the fundamental nature of these peculiar difficulties has led many eminent workers to suggest various basic modifications [14–19] in the CTEM.

Alternatively it has been argued [20–24] that perhaps the conventional formulation of the self-interaction or the expressions for the energy-momentum content of the EM fields are not fully justified, since these appear relativistically non-covariant. With the belief that the energy and momentum of the EM fields should alway behave like the components of a 4-vector under a Lorentz transformation, a modified definition of the energy-momentum density of EM fields *associated with electric charges*, seems to be gaining wider acceptance [4, 5, 25–28]. This modified definition in fact has been used [29] for 'explaining' the null-results of the famous Trouton–Noble experiment [30], where the conventional definition of the energy and momentum of electromagnetic fields paradoxically seems to predict a turning moment on a freely suspended charged capacitor, as observed from a relatively moving inertial frame of reference. More recently a number of arguments have appeared in the literature both for and against the modified definition [31–35]. Indeed the modified definition comprises the volume integrals of the erstwhile defined energy and momentum densities of fields, but now computed with respect to a 3D-space volume fixed in some specific inertial frame. This is also apparent from the explicit presence of u, the velocity of the specific inertial frame, in the modified definition. In essence, here one has to always first specify an inertial frame of reference, define the energy and momentum density of the EM fields in this particular frame, and then from the conditions of relativistic covariance can the energy and momentum density of the EM fields in all other inertial frames of reference be defined. This specification of an *initial* inertial frame (somewhat arbitrary, at least in the general case), almost goes against the spirit of the special theory of relativity. In fact by specifying a *different* initial inertial frame, one could arrive at a different value for the energy-momentum content of a *given* EM field in any inertial frame of reference. For the energy–momentum content of an EM field to have a proper *physical* meaning, within the framework of the special theory of relativity, it is certainly desirable that one should be able to define in principle, the energy and momentum densities in any inertial frame of reference merely in terms of the field values in that frame, without a need for referring to another inertial frame.

The 'troublesome factor of 4/3' in the electromagnetic mass of a spherical charge distribution, or some equivalent numerical factors for other charge distributions, arise in what is a pure electromagnetic description of pure electric charges, and therefore their explanation also must be found within the realm of the CTEM itself. Furthermore, for calculating the energy-momentum of exclusively the EM fields of a charge distribution, one should be able to do so without really worrying about the non-EM forces that may be holding the charges in place. Here the basic question is not about the description of an 'actual' elementary particle by a pure EM model, rather the question is about the mathematical self-consistency of the CTEM itself. We attempt to resolve it here by explicitly showing that all that is needed is to take into account *all* the work done by or against *all* the EM forces in arriving at that charge distribution. In that way we show that the earlier proposed modifications of the CTEM or the changes more recently suggested in the literature in the definition of energy-momentum densities of EM fields are not justified, and that the conventional formulation in the CTEM is fully consistent with the special theory of relativity. As we will see further, a full accounting of the work done by or against all the EM forces is necessary not only in the 'classical-electron' models but also in all other types of mascroscopic charge distributions in the CTEM, even in the case of a charged parallel plate capacitor where, while calculating the stored electromagnetic energy, we never bother about the non-EM forces that keep the charges from flying away from the plates. Although here we will consider only some definite simple charge distributions, yet the conclusions drawn are of the most general nature and thus applicable to any charge distribution in classical electromagnetism.

2. 'Classical-electron' model

Right at the outset we should emphasize that our intention here is not to argue for a model of an 'actual' electron using only classical electromagnetism, rather we are only trying to bring out the source of apparent discrepancies in the description of the charge distributions, such as considered previously by Abraham [9] and Lorentz [1, 11] for the 'electron' models, all strictly within 'classical' electromagnetism. In fact our arguments are valid for finite size charge distributions, even on macroscopic scales. First we will consider the model where the charge is distributed over a thin spherical shell (a surface-charge distribution), and then we will also examine the case of a volume-charge distribution within a solid sphere.

2.1. A uniformly charged spherical shell model

We assume here the charge to be distributed uniformly over a spherical shell of radius r. We assume that 'always', there is available an inertial frame of reference, called the rest-frame, in which the charge distribution remains rigidly spherical, i.e. there is no relative motion between various parts of the charge distribution in its instantaneous rest-frame. In that sense, the charge distribution follows a 'rigid motion' [7], and any acceleration of the system can be thought of as successive transitions of the system through a series of 'rest-frames', in each of which the charge distribution comes to rest momentarily.

Now there is a mutual force of electrostatic repulsion between various parts of this charge distribution, and each element feels an outward repulsive force of $2\pi\sigma^2$ per unit area [36] in the rest-frame, where σ is the surface charge density. But due to its spherical symmetry, *the net force on the 'rigid' shell is zero. The electric field is zero inside the shell and follows the inverse square law on the outside.* The total energy, U_0, in the electrostatic field, calculated from the volume integral of the field energy-density $E^2/8\pi$, is equal to the self-potential energy of the charged sphere, i.e.

$$U_0 = \int \frac{E^2}{8\pi}\,\mathrm{d}v = \frac{1}{2}\int\int \frac{\rho(x)\rho(x')}{|x-x'|}\,\mathrm{d}v\,\mathrm{d}v' = \frac{e^2}{2r} \tag{1}$$

here ρ represents the volume density of the charge distribution and $e = \int \rho\,\mathrm{d}v = 4\pi r^2\sigma$ is the total charge of the sphere. Here all volume integrals are in the rest-frame of the charge distribution.

With this field energy U_0, we can associate a mass, U_0/c^2, called the electromagnetic mass of the system. By definition the momentum of the system is zero in the rest-frame. Seen from another frame K, with respect to which the charge is moving with a velocity u along the x-axis, the energy and momentum of the system are different as compared to those measured in the rest-frame K'. Now two points need to be looked at carefully.

Firstly there is a Lorentz contraction of the sphere by a factor $\gamma = 1/\sqrt{1-u^2/c^2}$, along the direction of motion. This not only changes the shape of the charge distribution into an ellipsoid, as seen in K, but also the resultant surface charge density is no longer uniform (although the charge still remains uniformly distributed over a rigid sphere as seen in the rest-frame K'). The surface elements lying along the direction of motion have higher surface charge density due to Lorentz contraction, as seen in K, compared to those that are lying normal to the x-axis. This in turn causes a greater concentration of the lines of electric flux towards a plane normal to the direction of motion for the field of a moving charge. It should be emphasized that the Lorentz contraction is a real contraction in space [37, 38] and that the ellipsoid with a non-uniform surface charge density has a different self-potential energy from a uniformly charged spherical distribution. Actually work has been done during the Lorentz contraction against the forces of self-repulsion, and this excess energy has to be supplied by the very agency that is responsible for the state of 'rigid motion'. Of course the same excess energy also appears in the *electric* field of the moving charge. *This energy increment during* Lorentz contraction is over and above the increase in relativistic energy given by the usual relativistic transformation formulae, which are applicable to a neutral mass with no forces of repulsion within it. Secondly there is an excess momentum of the charge distribution over and above the usual relativistic momentum formula $\mathscr{E}u/c^2$, where $\mathscr{E}$ is the total energy of the system in frame K. This extra momentum is due to the fact that for this given charged-particle system, as it 'rigidly' moves along the x-axis (towards *right, say*), *there is a continuous flow of energy into* the system at its left-sided surface (due to the work being done at a rate $\boldsymbol{F}\cdot\boldsymbol{u}$, where $\boldsymbol{F}$ is the net force of electrostatic repulsion on the left-half of the charge distribution) and the same amount of energy is flowing *out* of the system at its right-sided surface. Although there is no net increase of the system energy due to these flows, there is a continuous transport of energy taking place from the left-sided surface to the right-sided surface within the system, and this forms a part of the total momentum of the system. Again this is in addition to the momentum of the system due to its overall bodily movement towards the right, such as in the case of motion of even a neutral mass.

We can calculate the energy $\mathscr{E}$ and momentum $\mathscr{P}$ of a moving charge distribution in the following way. Let us consider the change in energy of the system when it is taken from its state of rest to that of a finite velocity along the x-axis, as seen in an inertial frame K. The implied transition of the system through successive 'instantaneous' rest-frames would also mean, at each step, a change in Lorentz contraction of the system, as seen in K. The total increase in energy of the charged particle is given by

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t}=\frac{\mathrm{d}\mathscr{P}}{\mathrm{d}t}u+\frac{\mathrm{d}\mathscr{W}_{\mathrm{LC}}}{\mathrm{d}t} \tag{2}$$

here the first term on the right-hand side represents the gain in energy due to bodily acceleration of the system as a whole, and the second term is due to the work done against the forces of self-repulsion of the charge distribution during Lorentz contraction.

The rate of work done on the system during the Lorentz contraction can be easily calculated. All dimensions along the x-axis shrink at a rate $\mathrm{d}(\gamma^{-1})/\mathrm{d}t$. Consider two infinitesimal surface elements in the shape of circular rings, each of radius $r\sin\theta$ and of angular width $\mathrm{d}\theta$, separated by a distance $l=2r\cos\theta$ on two opposite sides of the spherical shell along the x-axis, as seen in the rest-frame K' (figure 1). Each ring has a surface area $\mathrm{d}S=2\pi r^2\sin\theta\,\mathrm{d}\theta$, and the outward force on each ring along the direction of their separation (x-axis) is $2\pi\sigma^2\cos\theta\,\mathrm{d}S$, as seen in K'. As seen in frame K, the x-component of force on each ring remains unaltered during a Lorentz transformation [39] and the distance between both rings contracts at a rate

$$\frac{\mathrm{d}l}{\mathrm{d}t}=2r\cos\theta\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\gamma}\right)$$

so that the rate of work done on these rings due to Lorentz contraction is given by

$$-4\pi\sigma^2 r\cos^2\theta\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\gamma}\right)\mathrm{d}S.$$

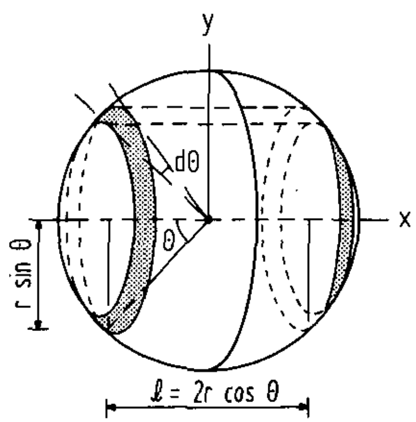

Figure 1. Geometry of the charged spherical shell, in its rest-frame.

After integration over the whole surface, we get the rate of total work being done on the charged sphere due to Lorentz contraction as

$$\frac{\mathrm{d}\mathscr{W}_{\mathrm{LC}}}{\mathrm{d}t} = -2\pi\sigma^2 V_0 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\gamma}\right) \tag{3}$$

where $V_0 = 4\pi r^3/3$ is the total volume of the sphere in its rest-frame.

The total momentum of the system in frame K includes the term due to energy flow associated with the EM forces within the system. Again consider two infinitesimal surface elements in the shape of rings, and the cross section of the system enclosed between them. At any instant when the system is moving with a velocity u, the work being done per unit time *on* the left ring is $2\pi\sigma^2 \cos\theta \, \mathrm{d}S\, u$ and at the same rate work is being done *by* the right ring, which lies at a distance $l = 2r\cos\theta/\gamma$. In other words there is an energy flow at a rate $4\pi\sigma^2 r \cos^2\theta \, \mathrm{d}Su/\gamma$, through a cross section $\mathrm{d}S\cos\theta$ of the system. Integration over the total cross section of the system yields a net momentum term due to this energy-flow

$$\frac{2\pi\sigma^2 V_0 u}{\gamma c^2}.$$

It should be emphasized that there is no 'instant' appearance of the energy from the left-sided surface to the right-sided one, rather there is a continuous flow of energy across its volume as seen in frame K. This energy flow is independent of the details of how the electromagnetic interaction takes place and is a simple consequence of the existence of the forces of mutual repulsion between various parts of the system and its linear motion as seen in frame K. Furthermore the contribution of this momentum term could be important even for non-relativistic velocities.

The total momentum of the system is therefore given as

$$\mathscr{P} = \left(\mathscr{E} + \frac{2\pi\sigma^2 V_0}{\gamma}\right)\frac{u}{c^2}. \tag{4}$$

Substituting (3) and (4) in (2) and writing $u/c = \beta$, after some simplifications we get,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathscr{E} + \frac{2\pi\sigma^2 V_0}{\gamma}\right) = \left(\mathscr{E} + \frac{2\pi\sigma^2 V_0}{\gamma}\right)\gamma^2\beta\frac{\mathrm{d}\beta}{\mathrm{d}t}.$$

Integrating with time and noting that $\mathscr{E}_{(t=0)} = U_0$, the self-potential energy for $\beta = 0$, we get

$$\mathscr{E} = \gamma(U_0 + 2\pi\sigma^2 V_0 \beta^2).$$

With the help of (1), we can write it as

$$\mathscr{E} = U_0\gamma\left(1 + \frac{\beta^2}{3}\right) = \frac{e^2\gamma}{2r}\left(1 + \frac{\beta^2}{3}\right) \tag{5}$$

Now substituting for $\mathscr{E}$ in (4) we also get

$$\mathscr{P} = (U_0 + 2\pi\sigma^2 V_0)\frac{\gamma u}{c^2}$$

or

$$\mathscr{P} = \frac{4}{3} U_0 \frac{\gamma u}{c^2} = \frac{4}{3}\left(\frac{e^2}{2r}\right)\frac{\gamma u}{c^2}. \tag{6}$$

Equations (5) and (6) are the long sought-after formulae for the electromagnetic energy and momentum of a moving charge. Note the presence of the factor 4/3 in (6) even for non-relativistic velocities.

Now we can also calculate volume integrals of the energy and momentum densities of the EM fields of a moving charge in frame K, from the standard formulae [26–28, 36] as,

$$\mathscr{E}_{\text{field}} = \int \frac{E^2 + B^2}{8\pi} \, \mathrm{d}v \qquad \mathscr{P}_{\text{field}} = \int \frac{\boldsymbol{E} \times \boldsymbol{B}}{4\pi c} \, \mathrm{d}v$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields of the uniformly moving charge, with $\boldsymbol{B} = \boldsymbol{\beta} \times \boldsymbol{E}$. In $\mathscr{P}_{\text{field}}$ the volume integral over only the x-component is non-zero because of the circular-cylindrical symmetry of the system about the x-axis.

These integrals can be evaluated more simply by making a change of variable into the coordinates of K' and using the transformation relation in fields and volume element between K and K',

$$E_{\parallel} = E'_{\parallel} \qquad E_{\perp} = \gamma E'_{\perp} \qquad \mathrm{d}v = \mathrm{d}v'/\gamma$$

to get [28]

$$\mathscr{E}_{\text{field}} = \gamma U_0 \left(1 + \frac{\beta^2}{3}\right) \qquad \mathscr{P}_{\text{field}} = \frac{4}{3} U_0 \frac{\gamma u}{c^2}$$

and these are exactly the expressions as obtained for the 'particle' picture of the charge distribution, namely (5) and (6).

2.2. A solid spherical charge distribution

Actually this case is essentially the same as that of the spherical shell model, differing only in the mathematical detail. If ρ is the uniform charge density, distributed in a spherical volume of radius R, then the radial force density due to the electrostatic field inside the sphere, in the rest-frame K', is given by

$$\rho E = \frac{4\pi}{3} \rho^2 r$$

for $r \leqslant R$.

Consider a spherical shell of a radius r and of a thickness $\mathrm{d}r$, and hence of a surface charge density $\rho \, \mathrm{d}r$ as seen in the rest-frame K'. The rate of work done against the electromagnetic forces during Lorentz contraction of this charged shell, as seen in K, is calculated to be

$$-\left(\frac{4\pi}{3}\rho\right)^2 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\gamma}\right) r^4 \, \mathrm{d}r.$$

An integration over the whole spherical volume gives the total rate of work being done against the Lorentz contraction as

$$\frac{\mathrm{d}\mathscr{W}_{\text{LC}}}{\mathrm{d}t} = -\frac{e^2}{5R} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\gamma}\right) \qquad (7)$$

where $e = 4\pi\rho R^3/3$ is the total charge of the system.

In the same manner the momentum component due to energy-flow caused by the forces of self-repulsion is calculated to be

$$\left(\frac{4\pi}{3}\rho R^3\right)^2 \frac{u}{\gamma c^2}\frac{1}{5R} = \frac{e^2}{5R}\frac{u}{\gamma c^2}.$$

Therefore the total momentum is written as

$$\mathscr{P} = \left(\mathscr{E} + \frac{e^2}{5R\gamma}\right)\frac{u}{c^2}. \tag{8}$$

Using (7) and (8) and proceeding from (2) as in the earlier case, we finally get

$$\mathscr{E} = \frac{3}{5}\frac{e^2}{R}\gamma\left(1+\frac{\beta^2}{3}\right) = U_0\gamma\left(1+\frac{\beta^2}{3}\right) \tag{9}$$

and

$$\mathscr{P} = \frac{4}{5}\frac{e^2}{R}\frac{u\gamma}{c^2} = \frac{4}{3}U_0\frac{\gamma u}{c^2} \tag{10}$$

where $U_0 = 3e^2/5R$ is the self-potential energy of the charge distribution in its rest-frame.

If we calculate the energy and momentum of the EM fields of this charge distribution, now moving with a uniform velocity u in the inertial frame K, we get formulae for $\mathscr{E}_{\text{field}}$ and $\mathscr{P}_{\text{field}}$ identical to those as calculated for the 'particle' picture of the charge distribution, namely (9) and (10).

Before proceeding further we would like to point out an interesting aspect of the energy distribution between electric and magnetic fields of a moving charge. If we calculate energies in the electric and magnetic fields separately, we find that in a semi-relativistic treatment (up to terms of order β^2), the energy in electric field represents the self-potential energy (internal energy!) of the charge distribution, while that in the magnetic field represents the kinetic energy of motion of the system. For example, the energy in the electric field of a moving charge, in frame K, is,

$$\mathscr{E}_{\text{el}} = \int \frac{E^2}{8\pi}\,dv = \gamma U_0\left(1-\frac{\beta^2}{3}\right) \simeq U_0\left(1+\frac{\beta^2}{6}\right).$$

Thus the energy in electric field of a moving charge is higher than that of a stationary charge by an amount $U_0\beta^2/6$. Actually this increase just reflects the gain in the self-potential energy of the charge distribution during the Lorentz contraction, also calculated respectively from (3) or (7) as,

$$\frac{U_0}{3}\left(1-\frac{1}{\gamma}\right) \simeq U_0\frac{\beta^2}{6}.$$

In fact a calculation for the electrostatic self-potential energy of a stationary ellipsoid, obtained from a spherical charge distribution through a uniform compression of its linear dimensions along the x-axis by a factor $\sqrt{1-\beta^2}$ (thus realizing exactly the same charge distribution as of our moving charge seen in frame K), gives [40]

$$U_{\text{ell}} = U_0\frac{\sin^{-1}\beta}{\beta} \simeq U_0\left(1+\frac{\beta^2}{6}\right)$$

where U_0 is the self-potential energy of the spherical charge distribution, i.e. for $\beta = 0$. This further confirms our statement about the electric field energy representing basically the self-potential energy even for a moving charge.

At the same time the energy in magnetic field is calculated to be

$$\mathscr{E}_{\text{mag}} = \int \frac{B^2}{8\pi} \, dv = \frac{2}{3} \gamma U_0 \beta^2$$

which is related to the momentum of the charge distribution ((6) or (10)) by $\mathscr{E}_{\text{mag}} = \mathscr{P}u/2$, which is exactly the expression for kinetic energy in classical mechanics. Thus the analogy usually drawn between the electric and magnetic field energies in an oscillatory electrical circuit on one side and the potential and kinetic energies of a mechanical harmonic oscillator on the other side [41], becomes more of a homology in the case of motion of a pure EM charge.

Thus we see that there is no conflict in the 'particle' picture for a charge distribution and its EM fields in the CTEM. The earlier confusion was caused by the fact that the usual Lorentz transformation formulae for energy-momentum were thought to be equally valid for a *pure* charge distribution. Of course we do not imply that now we can have a pure electromagnetic model for some elementary charged particle actually found in nature, e.g. an electron or a muon. A quantum electrodynamical treatment [42] (or perhaps something more exotic [43]) only might give a realistic model of an elemenentary charged particle. Our intention here has been only to show that for a *given* charge distribution, without actually worrying about any non-electromagnetic forces which may ultimately be needed to keep the charges in place, we do get a consistent picture of the energy and momentum of electromagnetic origin when we properly take into account *all* the work done by or against *all* the electromagnetic forces that prevail in the system. In that sense the CTEM is *complete* in itself and is a mathematically consistent theory, fully compatible with the special theory of relativity.

This becomes clearer from the fact that the type of effects we have discussed above need to be taken into account not just in the case of the spherical charge distribution models of the 'classical-electron', but in *any* charge distribution in the CTEM, otherwise we would always run into some inconsistencies during a Lorentz transformation of the electromagnetic field energy and momentum. To illustrate this point further we shall now consider the simple case of a charged parallel plate capacitor, where for example, while calculating the energy stored in the electromagnetic fields, we never pause to wonder what may be the (non-electromagnetic) forces that keep the charges from flying away from the plate surfaces.

3. A charged parallel plate capacitor

We shall consider the motion of a charged parallel plate capacitor in directions both normal to the plate surfaces as well as parallel to the plate surfaces. As we will see, the effects of a Lorentz transformation are quite different in the two cases, and in fact in the latter case the electromagnetic forces that are at work are not immediately obvious.

3.1. Motion normal to the plate surfaces

For simplicity we assume that the dimensions (a and b) of the capacitor plates are much larger than the plate separation, h, so that the electric fields within the capacitor can be considered, with negligible errors, to be the same as in the case of infinite plates. Let $+\sigma$ and $-\sigma$ be the surface charge densities on the two oppositely charged plates, then the electrostatic field is a constant, $4\pi\sigma$, in the region between the two plates, and is zero everywhere outside. The mutual force of attraction on each plate is $2\pi\sigma^2$ per unit area [36], and the electric potential energy U_0 accumulated in separating the two plates by a distance h is $2\pi\sigma^2 Ah$, where $A = ab$ is the surface area of each plate. Of course the volume integral of the energy density of the electrostatic field, $(4\pi\sigma)^2 Ah/8\pi$, is also the same as U_0.

Now if we see this system from an inertial frame K, with respect to which the charged capacitor is moving along the x-axis with a velocity u (figure 2), the electric field strength remains the same, $E = E' = 4\pi\sigma$ within the capacitor volume and zero outside. But as the whole system is Lorentz contracted by a factor γ along the x-axis, the total EM field energy as calculated in K is U_0/γ. The magnetic field is zero everywhere, therefore the field momentum is zero.

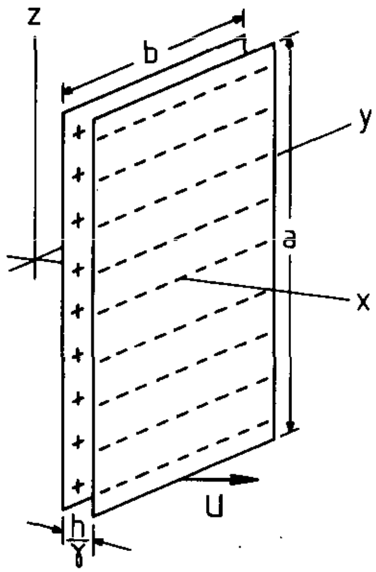

Figure 2. A parallel plate capacitor with a motion normal to the plate surfaces.

This appears to be in contradiction with the usual Lorentz transformation formulae where we would expect the energy of the moving system to have *increased* by a factor γ. Where has this energy disappeared?

Actually there is the force of attraction between the plates, and during a Lorentz contraction the system gives up energy to the very agency that is responsible for the state of 'rigid motion' [7]. Further there is a negative momentum component (i.e. in a direction opposite to the motion of the capacitor), due to the energy flow caused by the force of attraction between the moving plates.

Proceeding as in earlier cases, the rate of change of energy of the system is calculated to be

$$\frac{d\mathscr{E}}{dt} = \frac{d\mathscr{P}}{dt} u + 2\pi\sigma^2 V_0 \frac{d}{dt}\left(\frac{1}{\gamma}\right)$$

and total momentum of the system as

$$\mathscr{P} = \left(\mathscr{E} - \frac{2\pi\sigma^2 V_0}{\gamma}\right)\frac{u}{c^2}$$

where $V_0 = Ah$ is the volume of the region between the capacitor plates in the rest-frame. From these we get,

$$\mathscr{E} = \gamma(U_0 - 2\pi\sigma^2 V_0\beta^2) = \frac{U_0}{\gamma}$$

and

$$\mathscr{P} = \left(\frac{U_0}{\gamma} - \frac{2\pi\sigma^2 V_0}{\gamma}\right)\frac{u}{c^2} = 0$$

these being in perfect agreement with the values calculated above for the EM fields.

Thus we see that even in the simple case of a parallel plate capacitor, we get a consistent picture of the energy-momentum of the EM fields only by a proper accounting of the work done by the electromagnetic forces during the motion of the system.

3.2. *Motion along the plate surfaces*

We assume the plates to be lying in the x-y plane (figure 3). The electric field between the plates is parallel to the z-direction. The potential energy of the system as well as the energy in the electrostatic field is $U_0 = 2\pi\sigma^2 V_0$, in the rest-frame. But in frame K, with respect to which the capacitor plates are moving along the x-axis, the electric field between the plates goes up by a factor γ, thus there is an increase in the electric field energy-density in between the plates, by a factor γ^2. The total surface area of the plates, and hence also the volume enclosed by the capacitor plates, shrinks by a factor γ, there being no change in the plate separation. Therefore there is a net increase in the total electric field energy by a factor γ. This appears all right at a first look since the net force of attraction between the plates being in the z-direction, there is no work done by it due to the motion of the plates along x-axis, and thus the usual relativistic formula for the energy transformation seems applicable. But there is also a magnetic field ($\boldsymbol{B} = \boldsymbol{\beta} \times \boldsymbol{E}$) along the y-axis, in between the plates, as seen in K. The total energy in the EM fields therefore is $\gamma U_0(1+\beta^2)$, and also there is a net field momentum $2U_0\gamma u/c^2$, along the x-axis.

To seek the origin of this extra energy and momentum in the EM fields, it should be noted that there are electromagnetic forces of *repulsion* on charges *within* each plate, along its surface. We ignored these repulsive forces in the earlier case because no work is done by these forces for a motion normal to the plate surface, but in the present case of a motion parallel to the plate surface, work is done against these forces

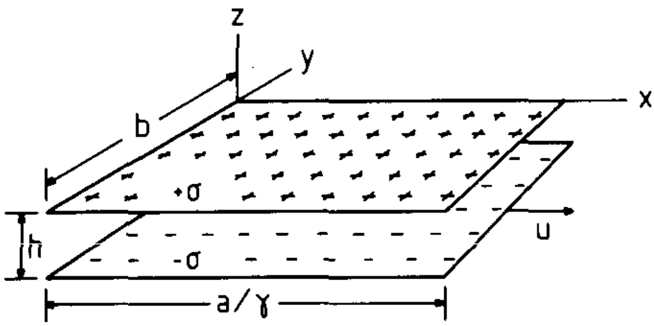

Figure 3. A parallel plate capacitor with a motion parallel to the plate surfaces.

during a Lorentz contraction. The forces are indeed small near the plate-centres and are appreciable only near the plate-edges, and it might seem that for the plate dimensions (a and b) large enough as compared to the plate separation (h), the effect of these forces should be negligible. But as we will see below, the amount of work done during a Lorentz contraction in this case also is proportional to the *total volume* enclosed within the capacitor, for $h \ll a, b$.

As all motions considered are along the x-axis, only the x-component of the forces of repulsion will be relevant for our purpose. Now the mutual electrostatic force of repulsion between two line charges, each with a linear charge density λ and of a length b, separated by a distance x is easily calculated to be

$$\frac{2\lambda^2}{x}(\sqrt{b^2+x^2}-x)$$

Accordingly the net force of repulsion on a line charge of linear charge density $\sigma\,dx$ lying at x, due to both plates (figure 4) is given by

$$2\sigma^2\,dx\left[\int_0^{2x-a} dx'\left(\frac{\sqrt{b^2+(x-x')^2}-(x-x')}{x-x'}\right)-\int_0^{2x-a} dx'\frac{(x-x')}{\sqrt{h^2+(x-x')^2}}\right.$$
$$\left.\times\left(\frac{\sqrt{b^2+h^2+(x-x')^2}-\sqrt{h^2+(x-x')^2}}{\sqrt{h^2+(x-x')^2}}\right)\right]$$

where the second integral term represents the x-component of the force of attraction on the line element at x due to the oppositely charged plate lying at a distance h below. Here we have taken the line element at x to be in the right-half of the plate, which experiences a net force towards the $+ve$ x-axis; the left-half of each plate would equally experience a net force along the $-ve$ x-axis. Further, only the portion of each plate lying between 0 and $2x-a$ contributes a net force at x, the force due to the remaining portion of each plate gets cancelled because of its symmetry about x.

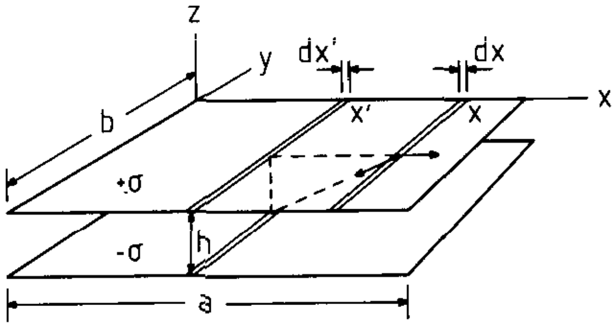

Figure 4. Geometry of the parallel plate capacitor in its rest-frame, for calculating the forces of self-repulsion within each plate of the capacitor.

Now as observed from an inertial frame K, with respect to which the capacitor moves towards the right (along $+ve$ x-axis) with a velocity u, the x-component of force on a charge element remains invariant. Therefore the rate of work being done against the forces of self-interaction, during Lorentz contraction of *both plates*, is written as (with a change of variable $x-x'=\xi$),

$$-4\sigma^2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\gamma}\right)\int_{a/2}^{a}\mathrm{d}x\,(2x-a)\int_{a-x}^{x}\mathrm{d}\xi\left\{\frac{\sqrt{b^2+\xi^2}-\xi}{\xi}-\frac{\xi}{h^2+\xi^2}(\sqrt{b^2+h^2+\xi^2}-\sqrt{h^2+\xi^2})\right\}$$

$$=-4\sigma^2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\gamma}\right)\int_{0}^{a}\mathrm{d}x\,(2x-a)\Bigg[(\sqrt{x^2+b^2}-x+\sqrt{x^2+h^2}-\sqrt{x^2+b^2+h^2})$$

$$-b\ln\left(\frac{\sqrt{x^2+b^2+h^2}-b}{\sqrt{x^2+h^2}}\frac{x}{\sqrt{x^2+b^2}-b}\right)\Bigg]$$

where $\gamma=1/\sqrt{1-u^2/c^2}$ is the Lorentz factor. An extra factor of 2 in the above expression has entered because work is done against Lorentz contraction of each plate.

With the help of the indefinite integrals,

$$\int\ln\left(\frac{\sqrt{x^2+b^2+h^2}-b}{\sqrt{x^2+h^2}}\right)\mathrm{d}x$$

$$=x\ln\left(\frac{\sqrt{x^2+b^2+h^2}-b}{\sqrt{x^2+h^2}}\right)+b\ln(\sqrt{x^2+b^2+h^2}-x)$$

$$+h\tan^{-1}\frac{bx}{h\sqrt{x^2+b^2+h^2}}$$

and

$$\int x\ln\left(\frac{\sqrt{x^2+b^2+h^2}-b}{\sqrt{x^2+h^2}}\right)\mathrm{d}x=\tfrac{1}{2}(x^2+h^2)\ln\left(\frac{\sqrt{x^2+b^2+h^2}-b}{\sqrt{x^2+h^2}}\right)-\frac{b}{2}\sqrt{x^2+b^2+h^2}$$

and after some lengthy calculations, we finally get the following expression for the rate of work done against Lorentz contraction of the system,

$$-4\sigma^2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\gamma}\right)\Bigg[\frac{2h^2}{3}(\sqrt{a^2+h^2}-h+\sqrt{b^2+h^2}-\sqrt{a^2+b^2+h^2})$$

$$-\frac{b^2}{3}(\sqrt{b^2+h^2}-b+\sqrt{a^2+b^2}-\sqrt{a^2+b^2+h^2})$$

$$+\frac{a^2}{6}(\sqrt{a^2+h^2}-a+\sqrt{a^2+b^2}-\sqrt{a^2+b^2+h^2})$$

$$+\frac{ab^2}{2}\ln\left(\frac{\sqrt{a^2+b^2+h^2}-a}{\sqrt{b^2+h^2}}\frac{b}{\sqrt{a^2+b^2}-a}\right)$$

$$-\frac{ah^2}{2}\ln\left(\frac{\sqrt{a^2+b^2+h^2}-a}{\sqrt{b^2+h^2}}\frac{h}{\sqrt{a^2+h^2}-a}\right)$$

$$-bh^2\ln\left(\frac{\sqrt{a^2+b^2+h^2}-b}{\sqrt{a^2+h^2}}\frac{h}{\sqrt{b^2+h^2}-b}\right)+abh\tan^{-1}\frac{ab}{h\sqrt{a^2+b^2+h^2}}\Bigg].$$

We can expand this complicated-looking expression in terms of an ascending power series in h/a, h/b, $h/\sqrt{a^2+b^2}$ as

$$-4\sigma^2abh\left[\frac{\pi}{2}+\mathrm{O}\left(\frac{h}{a},\frac{h}{b},\frac{h}{\sqrt{a^2+b^2}}\right)\right]\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\gamma}\right)$$

where O(...) represents the first- and higher-order power series terms in h/a, h/b, etc. Therefore for $h \ll a, b$, this reduces to

$$-2\pi\sigma^2 V_0 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\gamma}\right)$$

where $V_0 = abh$ is the volume enclosed in between the capacitor plates in the rest-frame.

In a similar manner the component of momentum due to energy flow associated with the forces of self-interaction in the system is calculated to be

$$\frac{2\pi\sigma^2 V_0 u}{\gamma c^2}$$

Using these last two expressions and proceeding in the same manner as in the earlier cases, we get

$$\mathscr{E} = \gamma U_0(1+\beta^2) \qquad \text{and} \qquad \mathscr{P} = 2U_0\gamma\frac{u}{c^2}$$

which are exactly the expressions obtained for the energy and momentum of the EM fields, in frame K.

4. Discussion

We have seen that with a proper accounting of all contributions by the EM forces in the system, the relativistic energy and momentum of a macroscopic charge distribution turns out to be the same as calculated from the conventional expressions for the energy–momentum density of its EM fields, although sometimes the effect of these forces may not be so obvious, as for example in the case of a charged capacitor moving parallel to its plate surfaces. In particular it was seen that the potential energy of a charge distribution changes, due to Lorentz contraction of the system, when it is set in motion. This change must reflect in the electric field energy since that basically represents the potential energy of the charge distribution. Further, there is a momentum associated with the energy flow due to forces of EM interaction in a moving charge system, and this also should show up in the EM fields of the system. As we have shown, the conventional expressions for the energy and momentum of the EM fields do comprise these contributions. Thus the notion that the energy–momentum of EM fields should behave as a 4-vector, even in the presence of charges, is ill-founded, since that ignores the additional contribution of the EM forces to the energy and momentum, when the system is set in motion. The changes suggested in the literature in the definition of energy–momentum of EM fields, so as to make them *always* transform as a 4-vector under a Lorentz transformation, are not justified, since that would amount to selectively excluding some effects of the EM interaction from the EM fields. It is only by including the contribution of *all* EM interactions in the energy–momentum of EM fields, we can hope to maintain a consistent picture throughout, and that is done precisely by the conventional definition.

Since no static system could be 'purely electromagnetic' in nature, a question may arise about the transformation properties of the energy and momentum of a complete system. A *static* configuration of any electromagnetic system in its rest-frame implies that there are forces of constraint in the system which necessarily have to be equal and opposite to the forces of electromagnetic interaction everywhere, irrespective of the ultimate nature of the forces of constraint. For example in the case of a charged parallel plate capacitor, the forces of constraint would not only maintain the fixed separation between the capacitor plates in spite of the force of mutual attraction between them, but would also keep the charges confined to their respective locations on the plate surfaces in spite of the net force of electrostatic repulsion on them. Now as seen from another inertial frame, with respect to which the charged system is in motion, the work done by the forces of constraint during a Lorentz contraction and the momentum associated with the energy flow due to forces of constraint would be equal and opposite to those corresponding to the forces of electromagnetic origin, since the two forces are everywhere equal and opposite. An inclusion of the contribution of the terms due to forces of constraint would therefore cancel the corresponding terms arising due to the forces of electromagnetic origin (for example, the second term on the right-hand side of (2) and (4)) in the energy and momentum of the moving system. As a result, the energy-momentum of the *total system* would behave as a 4-vector under a Lorentz transformation. This in fact is the key factor in a proper resolution [44] of the paradox associated with the null-results of the Trouton–Noble experiment, where the energy and momentum of the total system remain independent of the orientation of the freely suspended charged capacitor, with respect to all inertial frames of reference. Thus we see that the energy and momentum of a 'real' charged particle, which includes contributions both of the electromagnetic self-interaction and of the forces of constraint, would transform as a 4-vector, while the energy and momentum of its EM fields, which can represent only the 'pure electromagnetic part', would not transform as a 4-vector.

5. Conclusions

We have shown that the energy and momentum of a 'pure electromagnetic' charged system, with net electromagnetic forces between its parts, does not behave as a 4-vector under a Lorentz transformation. Accordingly the energy and momentum of the electromagnetic fields, which represent just the 'pure electromagnetic part' of a real charged system, would also not behave as a 4-vector under a Lorentz transformation, in the presence of electromagnetic forces. In that respect, therefore, there is no conflict between the CTEM and the theory of relativity and any lingering doubts about the mathematical self-consistency of the CTEM are removed. On the other hand, the modified definition, as suggested in the literature for the energy and momentum of electromagnetic fields so as to make them always transform as components of a 4-vector, is not justified since it is only the total energy and momentum (including that of the forces of constraint) of a system which would transform as a 4-vector.

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References

[1] Lorentz H A 1904 *Proc. Acad. Sci. Amst.* **6** 809 (Reprinted in: Lorentz H A, Einstein A, Minkowski H and Weyl H 1952 *The Principle of Relativity* (New York: Dover) p 9)
[2] Einstein A 1905 *Ann. Phys.* **17** 891 (Reprinted in: Lorentz H A, Einstein A, Minkowski H and Weyl H 1952 *The Principle of Relativity* (New York: Dover) p 37)
[3] Feynman R P, Leighton R B and Sands M 1964 *The Feynman Lectures on Physics* vol II (Reading, MA: Addison-Wesley) ch 28
[4] Rohrlich F 1965 *Classical Charged Particles* (Reading, MA: Addison-Wesley)
[5] Butler J W 1969 *Am. J. Phys.* **37** 1258
[6] Pais A 1972 *Aspects of Quantum Theory* eds A Salam and E Wigner (Cambridge: Cambridge University Press) p 79
[7] Pearle P 1982 *Electromagnetism—Paths to Research* ed D Teplitz (New York: Plenum) p 211
[8] Thomson J J 1881 *Phil. Mag.* [V] **11** 229
[9] Abraham M 1903 *Ann. Phys.* **10** 105
[10] Thomson J J 1897 *Phil. Mag.* [V] **44** 293
[11] Lorentz H A 1952 *The Theory of Electron* 2nd edn (New York: Dover)
[12] Schott G A 1912 *Electromagnetic Radiation* (Cambridge: Cambridge University Press)
[13] Poincaré H 1906 *Rend. Circ. Mat. Palermo* **21** 129 (Engl. transl. with modern notation in Schwartz H M 1972 *Am. J. Phys.* **40** 860)
[14] Mie G 1912 *Ann. Phys.* **37** 511
[15] Born M and Infeld L 1934 *Proc. R. Soc.* A **144** 425
[16] Bopp F 1940 *Ann. Phys.* [5] **38** 345
[17] Dirac P A M 1938 *Proc. R. Soc.* A **167** 148
[18] Fokker A D 1929 *Z. Phys.* **58** 386
[19] Wheeler J A and Feynman R P 1949 *Rev. Mod. Phys.* **21** 425
[20] Fermi E 1922 *Phys. Zeits.* **23** 340
[21] Mandel H 1926 *Z. Phys.* **39** 40
[22] Wilson W 1936 *Proc. Phys. Soc.* **48** 736
[23] Kwal B 1949 *J. Phys. Radium* **10** 103
[24] Rohrlich F 1960 *Am. J. Phys.* **28** 639
[25] Aharoni J 1965 *The Special Theory of Relativity* (Oxford: Clarendon)
[26] Jackson J D 1975 *Classical Electrodynamics* (New York: Wiley)
[27] Panofsky W K H and Phillips M 1962 *Classical Electricity and Magnetism* (Reading, MA: Addison-Wesley)
[28] Konopinski E J 1981 *Electromagnetic Fields and Relativistic Particles* (New York: McGraw-Hill)
[29] Butler J W 1968 *Am. J. Phys.* **36** 936
[30] Trouton F T and Noble H R 1903 *Phil. Trans. R. Soc.* A **202** 165
[31] Boyer T H 1982 *Phys. Rev.* D **25** 3246
[32] Rohrlich F 1982 *Phys. Rev.* D **25** 3251
[33] Bialynicki-Birula I 1983 *Phys. Rev.* D **28** 2114
[34] Boyer T H 1985 *Am. J. Phys.* **53** 167
[35] Campos I and Jiménez J L 1986 *Phys. Rev.* D **33** 607
[36] Purcell E M 1965 *Electricity and Magnetism-Berkeley Physics Course* vol II (New York: McGraw-Hill)
[37] Tolman R C 1934 *Relativity, Thermodynamics and Cosmology* (Oxford: Clarendon)
[38] Arzeliés H 1966 *Relativistic Kinematics* (Oxford: Pergamon)
[39] Kittel C, Knight W D and Ruderman M A 1965 *Mechanics - Berkeley Physics Course* vol I (New York: McGraw-Hill)
[40] Searle G F C 1897 *Phil. Mag.* [V] **44** 329
[41] Crawford Jr F C 1968 *Waves - Berkeley Physics Course* vol III (New York: McGraw-Hill)
[42] Heitler W 1954 *The Quantum Theory of Radiation* (Oxford: Clarendon)
[43] Dehmelt H 1990 *Rev. Mod. Phys.* **62** 525
[44] Singal A K 1992 *Am. J. Phys*, submitted